Final Exam — Analysis (WPMA14004)

Tuesday 20 June 2017, 9.00h–12.00h

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. Provide clear arguments for all your answers: only answering "yes", "no", or "42" is not sufficient. You may use all theorems and statements in the book, but you should clearly indicate which of them you are using.
3. The total score for all questions equals 90. If p is the number of marks then the exam grade is $G = 1 + p/10$.

Problem 1 (15 points)

Assume that the sets $A, B \subset \mathbb{R}$ are both nonempty and bounded below. Prove that

$$\inf(A \cup B) = \min\{\inf A, \inf B\}.$$

Hint: first explain that it suffices to consider only the case $\inf A \leq \inf B$.

Problem 2 (5 + 6 + 4 = 15 points)

Assume that (a_n) and (b_n) are positive sequences such that

$$\lim \frac{a_n}{b_n} = c > 0.$$

Prove the following statements:

(a) For all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$n \geq N \quad \Rightarrow \quad (c - \epsilon)b_n < a_n < (c + \epsilon)b_n.$$

(b) The series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{n=1}^{\infty} b_n$ converges.

(c) The series $\sum_{n=1}^{\infty} \sin(\frac{1}{n})$ diverges. Hint: what is $\lim_{x\to 0} \frac{\sin x}{x}$?

Problem 3 (8 + 7 = 15 points)

(a) Prove that if $A \subseteq \mathbb{R}$ is compact, then for each $\epsilon > 0$ there exist finitely many points $a_1, \ldots, a_n \in A$ such that

$$A \subset V_\epsilon(a_1) \cup V_\epsilon(a_2) \cup \cdots \cup V_\epsilon(a_n).$$

Recall that $V_\epsilon(a) = (a - \epsilon, a + \epsilon)$.

(b) Show that for the set $A = \left\{ \frac{1}{n} \; : \; n \in \mathbb{N} \right\}$ the converse of part (a) does *not* hold.

Problem 4 (4 + 6 + 5 = 15 points)

(a) State the Mean Value Theorem.

(b) Use the Mean Value Theorem to prove that for all $a > 0$ the function $f(x) = \ln(x)$ is uniformly continuous on $[a, \infty)$.

(c) Is $f(x) = \ln(x)$ also uniformly continuous on $(0, \infty)$?

Problem 5 (5 + 5 + 5 = 15 points)

Consider the following sequence of functions:

$$f_n : [0,1] \to \mathbb{R}, \qquad f_n(x) = \begin{cases} 2nx & \text{if } 0 \leq x < 1/2n, \\ 2 - 2nx & \text{if } 1/2n \leq x < 1/n, \\ 0 & \text{if } 1/n \leq x \leq 1. \end{cases}$$

(a) Compute the pointwise limit of (f_n) for all $x \in [0,1]$.

(b) Is the convergence uniform on $[0,1]$?

(c) Is the convergence uniform on $[\frac{1}{2}, 1]$?

Problem 6 (6 + 4 + 5 = 15 points)

Define the function $f : [1,2] \to \mathbb{R}$ given by $f(x) = 1/x$. Consider for $n \in \mathbb{N}$ a partition P of the interval $[1,2]$ which is given by the points

$$x_k = \frac{n+k}{n}, \quad k = 0, \ldots, n.$$

(a) Compute the upper sum $U(f, P)$.

(b) Compute the lower sum $L(f, P)$.

(c) Prove that f is integrable on $[1,2]$. Use an ϵ-argument!

End of test (90 points)

Solution of Problem 1 (15 points)

Without loss of generality we may assume that $\inf A \leq \inf B$. Otherwise we just exchange the names of the sets A and B.

An alternative argument is that the case $\inf B \leq \inf A$ has a similar proof since the set A and B appear in the formula in a symmetric way (i.e., interchanging the roles of A and B gives the same formula).
(5 points)

Therefore, we need to prove that $\inf(A \cup B) = \inf A$. To that end, we need to prove two things:

(i) $\inf A$ is a lower bound for $A \cup B$;

(ii) any other lower bound ℓ of $A \cup B$ satisfies $\ell \leq \inf A$.

Alternatively, we can prove that any number greater than $\inf A$ is no longer a lower bound of $A \cup B$.

Let $x \in A \cup B$ be arbitrary, then $x \in A$ or $x \in B$. Therefore, $\inf A \leq x$ or $\inf B \leq x$. Since $\inf A \leq \inf B$ it follows that $\inf A \leq x$ for all $x \in A \cup B$. We conclude that $\inf A$ is a lower bound for the set $A \cup B$.
(5 points)

Let ℓ be any lower bound for $A \cup B$. Since $\ell \leq x$ for all $x \in A \cup B$ it follows in particular that $\ell \leq x$ for all $x \in A$. Since $\inf A$ is the greatest lower bound of A it follows that $\ell \leq \inf A$ which also shows that $\inf A$ is the greatest lower bound of $A \cup B$.
(5 points)

Alternative argument. Let $\epsilon > 0$ be arbitrary then there exists an element $x \in A$ such that $x < \inf A + \epsilon$. This means that $\inf A + \epsilon$ is not a lower bound for A. Since $A \subset A \cup B$ it follows that $\inf A + \epsilon$ cannot be a lower bound for $A \cup B$. We conclude that $\inf A$ is the greatest lower bound of $A \cup B$.
(5 points)

Solution of Problem 2 (5 + 6 + 4 = 15 points)

(a) By definition of the statement $\lim(a_n/b_n) = c$ it follows that for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$n \geq N \quad \Rightarrow \quad \left|\frac{a_n}{b_n} - c\right| < \epsilon.$$

(3 points)

The latter inequality can be rewritten as

$$-\epsilon < \frac{a_n}{b_n} - c < \epsilon.$$

and rearranging terms gives

$$(c - \epsilon)b_n < a_n < (c + \epsilon)b_n.$$

(2 points)

(b) Let $\epsilon = \frac{1}{2}c$ (any $0 < \epsilon < c$ works) and let $N \in \mathbb{N}$ be as in part (a).

(i) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} 2a_n/c$ converges as well by the Algebraic Limit Theorem for series.

Since $b_n < 2a_n/c$ for all $n \geq N$ it follows by the Comparison Test that $\sum_{n=1}^{\infty} b_n$ converges as well. (Note that the first N terms do not matter for convergence.)
(3 points)

(ii) Conversely, if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} \frac{3}{2}cb_n$ converges as well by the Algebraic Limit Theorem for series.

Since $a_n < \frac{3}{2}cb_n$ for all $n \geq N$ it follows by the Comparison Test that $\sum_{n=1}^{\infty} a_n$ converges as well.
(3 points)

(c) Using the standard limit $\lim_{x\to 0} \sin(x)/x = 1$ it follows that with $a_n = \sin(1/n)$ and $b_n = 1/n$ we get $c = \lim a_n/b_n = 1$.
(2 points)

The series $\sum_{n=1}^{\infty} b_n$ is the harmonic series and hence diverges. By part (b) it then follows that the series $\sum_{n=1}^{\infty} a_n$ diverges as well.
(2 points)

Solution of Problem 3 (8 + 7 = 15 points)

(a) Let $\epsilon > 0$ be arbitrary. For each $a \in A$ the set $V_\epsilon(a)$ is open.
(2 points)

Note that $A \subset \cup_{a\in A} V_\epsilon(a)$, which means that the collection $\{V_\epsilon(a) : a \in A\}$ is an open cover for A.
(2 points)

Since A is compact any open cover has a finite subcover. In particular, this means that A can be covered by finitely many of the sets $V_\epsilon(a)$. Hence, there exist finitely many points $a_1, \ldots, a_n \in A$ such that

$$A \subset V_\epsilon(a_1) \cup V_\epsilon(a_2) \cup \cdots \cup V_\epsilon(a_n).$$

(4 points)

(b) First note that the set $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ is not compact since it is not closed. Indeed, 0 is a limit point of A which is not contained in A.
(3 points)

Let $\epsilon > 0$ be arbitrary, and choose $n_0 \in \mathbb{N}$ such that $1/n_0 < \epsilon$. Then $0 \in V_\epsilon(1/n_0)$ and the set A only has finitely many elements outside $V_\epsilon(1/n_0)$. Indeed,

$$\frac{1}{n} > \frac{1}{n_0} + \epsilon \quad \Rightarrow \quad n < \frac{n_0}{1 + \epsilon n_0}.$$

This proves that the noncompact set A can still be covered by finitely many of the open sets $V_\epsilon(1/n)$.
(4 points)

Solution of Problem 4 (4 + 6 + 5 = 15 points)

(a) If $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b), then there exists a point $c \in (a,b)$ where

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

(4 points)

(b) Let $x, y \in [a, \infty)$ be fixed and $x \neq y$; without loss of generality we may assume that $x < y$. The function f is continuous on $[x,y]$ and differentiable on (x,y) which means that the Mean Value Theorem can be applied. Hence, there exists $c \in (x,y)$ such that

$$\ln(x) - \ln(y) = \ln'(c)(x-y) = \frac{x-y}{c}.$$

Since $c > a$ it follows that

$$|\ln(x) - \ln(y)| = \frac{|x-y|}{c} \leq \frac{|x-y|}{a}.$$

(3 points)

Now let $\epsilon > 0$ be arbitrary and choose $\delta \leq a\epsilon$. Then

$$|x-y| < \delta \quad \Rightarrow \quad |\ln(x) - \ln(y)| \leq \frac{|x-y|}{a} < \frac{\delta}{a} \leq \epsilon,$$

which shows that $f(x) = \ln(x)$ is uniformly continuous on $[a, \infty)$.
(3 points)

(c) No, f is not uniformly continuous on $(0, \infty)$. To see this, take the sequences $x_n = e^{-n}$ and $y_n = e^{-(n+1)}$. Then $|x_n - y_n| \to 0$, but $\big|\ln(x_n) - \ln(y_n)\big| = 1$ for all $n \in \mathbb{N}$. By the sequential criterion for the absence of nonuniform continuity it follows that $f(x) = \ln(x)$ is not uniformly continuous on $(0, \infty)$.
(5 points)

Alternative argument. Let $\epsilon_0 = 1$ and choose $\delta > 0$ arbitrary. Choose $0 < y < \delta/(e-1)$ and $x = ey$, then

$$|x-y| = x - y = ey - y = (e-1)y < \delta,$$

but

$$|\ln(x) - \ln(y)| = \ln\left(\frac{x}{y}\right) = 1 = \epsilon_0,$$

which shows that $f(x) = \ln(x)$ is not uniformly continuous on $(0, \infty)$.
(5 points)

Solution of Problem 5 (5 + 5 + 5 = 15 points)

(a) Note that $f_n(0) = 0$ for all $n \in \mathbb{N}$ which implies that $\lim f_n(x) = 0$ for $x = 0$.
(2 points)

If $0 < x \leq 1$ then $f_n(x) = 0$ for all $n \geq 1/x$, which implies that $\lim f_n(x) = 0$ as well. Hence, the pointwise limit of (f_n) is the zero function.
(3 points)

(b) *Solution 1.* Note that $f_n(1/2n) = 1$ for all $n \in \mathbb{N}$, which implies that

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| = 1 \quad \text{for all} \quad n \in \mathbb{N}.$$

This implies that

$$\lim_{n\to\infty} \left(\sup_{x\in[0,1]} |f_n(x) - f(x)| \right) = 1 \neq 0.$$

Hence, the convergence is not uniform on $[0, 1]$.
(5 points)

Solution 2. If the convergence were uniform on $[0, 1]$, then for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$n \geq N \quad \Rightarrow \quad |f_n(x) - f(x)| < \epsilon \quad \text{for all} \quad x \in [0, 1].$$

In particular, this should hold for $0 < \epsilon < 1$. However, taking $x = 1/2n$ violates this definition. Hence, the convergence is not uniform on $[0, 1]$.
(5 points)

(c) *Solution 1.* Note that $f_n(x) = 0$ on $[\frac{1}{2}, 1]$ for all $n \geq 2$, which implies that

$$\lim_{n\to\infty} \left(\sup_{x\in[\frac{1}{2},1]} |f_n(x) - f(x)| \right) = 0,$$

which proves that the convergence is uniform on $[\frac{1}{2}, 1]$.
(5 points)

Solution 2. Let $\epsilon > 0$ be arbitrary and take $N = 2$. Then

$$n \geq N \quad \Rightarrow \quad |f_n(x) - f(x)| = f_n(x) = 0 < \epsilon \quad \text{for all} \quad x \in [\tfrac{1}{2}, 1],$$

which proves that the convergence is uniform on $[\frac{1}{2}, 1]$.
(5 points)

Solution of Problem 6 (6 + 4 + 5 = 15 points)

(a) Since the function is *decreasing* it follows that

$$M_k = \sup\{f(x) \,:\, x \in [x_{k-1}, x_k]\} = f(x_{k-1}) = \frac{1}{x_{k-1}} = \frac{n}{n+k-1}.$$

(2 points)

Furthermore, for all $k = 1, \dots, n$ we have that

$$x_k - x_{k-1} = \frac{n+k}{n} - \frac{n+k-1}{n} = \frac{(n+k)-(n+k-1)}{n} = \frac{1}{n}.$$

(2 points)

Hence, the upper sum of f with respect to the partition P is given by

$$U(f,P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) = \sum_{k=1}^{n} \frac{1}{n+k-1}.$$

(2 points)

(b) Since the function is *decreasing* it follows that

$$m_k = \inf\{f(x) \,:\, x \in [x_{k-1}, x_k]\} = f(x_k) = \frac{1}{x_k} = \frac{n}{n+k}.$$

(2 points)

Hence, the lower sum of f with respect to the partition P is given by

$$L(f,P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}) = \sum_{k=1}^{n} \frac{1}{n+k}.$$

(2 points)

(c) Note that the difference between the upper and lower sum is a telescoping sum:

$$U(f,P) - L(f,P) = \sum_{k=1}^{n} \left(\frac{1}{n+k-1} - \frac{1}{n+k} \right) = \frac{1}{n} - \frac{1}{2n} = \frac{1}{2n}.$$

(3 points)

Now, let $\epsilon > 0$ be arbitrary, and choose $n \in \mathbb{N}$ such that $1/n < 2\epsilon$. Then

$$U(f,P) - L(f,P) < \epsilon,$$

which proves that f is integrable.
(2 points)